On Three Septic Surfaces.

By John E. Hill.

If, in the general cubo-cubic transformation between two spaces,* we cause the principal sextic of one space to degenerate into a twisted quintic of deficiency 2 , C'_5, and into a right line, C'_1, meeting C'_5 twice, to the general cubic surface, S'_3, upon which C'_1 lies, there will correspond, in the second space, a septic surface, $S^{(7)}_{1^3,\,5^2}$, upon which C_1 is triple and C_5 is double. If, however, the principal sextic of the first space breaks up into a twisted quartic of the second kind, C'_4, and into a conic, C'_2, meeting C'_4 four times, to the general cubic surface, S'_3, passed through C'_2, there will correspond, in the second space, a septic surface, $S^{(7)}_{2^3,\,4^2}$, possessing C_4 doubly and C_2 triply. If, however, finally, the principal sextic of the first space degenerates completely, to the general cubic, S'_3, passed through the two transversals, A'_1, B'_1, and one line, C'_1, of the remaining ingredients, C'_1, D'_1, E'_1, F'_1, there will correspond, in the second space, a septic surface, $S^{(7)}_{1^3,\,1^3,\,1^3,\,1^2,\,1^2,\,1^2}$, possessing A_1, B_1, C_1 triply and D_1, E_1, F_1 doubly. It is the purpose of this paper to study some of the properties of these three septic surfaces by means of their plane representations.

The general plane section of *the surface*, $S^{(7)}_{1^3,\,1^3,\,1^3,\,1^2,\,1^2,\,1^2}$, is imaged, in its simplest plane representation, by a quartic curve passing through nine simple fundamental points; this surface, therefore, possesses only the nine lines whose images are these points and only the 36 conics whose images are those lines of the representative plane which join these points, two by two. Three of these conics, however, are the double lines possessed by the surface; for, if we number the fundamental points 1, 2, 3, 4, 5, 6, 7, 8, 9, the lines, say, 89, 97, 78 are the images of A_1, B_1 and C_1 respectively; so that the surface possesses, in fact, but 33 proper conics. There are 9 sheaves and 126 isolated examples of nodal

* See the article, "On Quintic Surfaces," in the Mathematical Review, vol. 1, No. 1.

cubics upon the surface, imaged, respectively, by the 9 sheaves of lines centering at the fundamental points and by the conics determined by the fundamental points taken five by five; of these last cubics, however, those imaged by the conics 1 2 3 4 5, 1 2 3 4 6 and 5 6 7 8 9 are the triple lines, A_1, B_1 and C_1.

By close study of the relations existing between the images in the representative plane, we see that no two lines of the surface meet, while each line meets two conics each once, to which, consequently, it is coordinated. On the other hand, each conic meets two lines in such a way that, as mentioned above, to each line there is coordinated two conics. Two conics meet or do not meet according as they do not or do have a common coordinated line. There are, moreover, three pairs of conics in which the ingredients of each pair meet twice. To make this last fact clear, we must first consider the relations that exist between the simple and multiple lines, and the multiple lines and the conics of the surface.

Each double line is met by two and but two simple lines, and this in such a way that there are six of the nine simple lines of the surface that do not meet a double line. On the other hand, five simple lines meet each triple line and in such a way that there is no simple line that does not meet a triple line. Four simple lines are common to A_1 and B_1, one common to A_1 and C_1 and one common to B_1 and C_1. Consequently, of these nine simple lines, each one of six meets two triple lines while each of the remaining three meets one triple and two double lines. Since A_1 and C_1 determine a plane, the line common to both is the residue of the section of $S^{(7)}$ determined by that plane; and similarly in the case of B_1 and C_1.

Each double line is met once by 21 conics and not at all by the other 12. Of these 21, however, there are 6 that still meet a double line twice; for, since A_1 and B_1 each meet D_1, E_1 and F_1, there are six planes thus determined whose residue sections of $S^{(7)}$ are conics; two of these six conics meet D_1 twice, two meet E_1 twice and two meet F_1 twice and in such a way that no two of them are common to any two of the three lines, D_1, E_1, F_1; these three pairs of conics are imaged, respectively, by 67 and 57, 68 and 58, and 69 and 59. Each triple line is not met at all by 10 conics (counting the double lines), is met once by 20 and twice by six; and there is no conic that does not meet a triple line. Moreover, of the 33 proper conics, there are 12 that are met twice by a triple line while the remaining 21 are met once. This, therefore, leaves but 21 isolated planes, whose sections consist of conics and quintics.

Each of the 12 conics determines, with the triple line that meets it twice, a plane whose residue is still a conic. The 12 conics, therefore, must, apparently, group themselves in pairs and form with the triple lines six plane sections of $S^{(7)}$. In fact, however, we easily see, from the plane projection, that but three pairs are so formed, the remaining six conics each pairing with a double line to make up those sections with the triple lines that were mentioned in the last paragraph. So that, counting the double lines as conics, we see that 15 of the 36 conics group themselves in nine pairs (each double line entering as ingredient into two such pairs) in such a way that there are three of such planes through each triple line. Excluding the double lines, the 12 conics form three pairs and two triplets; each pair forms with C_1 a complete plane section, while neither ingredient of any pair meets A_1 nor B_1, but each meets each double line once. The two triplets are composed of conics that each, in the one case, meets A_1 twice and B_1 once and, in the other case, meets A_1 once and B_1 twice; not one of such conics meets C_1, but each meets one double line once.

The remaining 21 conics group themselves into an octuplet, a 12-tuplet and an isolated case. The octuplet is composed of conics that meet two triple and the three double lines each once and, further, divides itself into two quadruplets according as to whether A_1 or B_1 is the second triple line that is met. The 12-tuplet is distinguished by conics that meet the three triple lines and one double line each once and, consequently, still separates itself into three quadruplets according to which double line it is that is met. Altogether, therefore, the 33 conics group themselves into an isolated case, three pairs, two triplets and five quadruplets. The isolated case is that conic that meets A_1, B_1, D_1, E_1 and F_1 each once, without meeting C_1.

Of nodal cubics, which include both plane and twisted curves, we have already noted nine sheaves and 126 isolated cases (including the three triple lines). In each sheaf there are eight cubics that break up into a line and a conic; such are imaged by the lines of the sheaf that pass through any one of the remaining eight fundamental points. In three of such last cases, the conic becomes a double line and the planes are, in fact, identical with the planes of the sheaf of planes through that double line, being, really, planes in which the residue quintic degenerates into a quartic and a right line, two such degenerations taking place in each sheaf.

Every cubic of the nine sheaves is met once by one and but one line, except

that in each sheaf there is one cubic that is met three times by such line, viz. the cubic of that plane in which the residue quartic breaks up into a cubic and a line. This will become apparent when such quartics are considered. Each cubic of these nine sheaves fails to meet eight conics while it meets each of the remainder once. No two cubics of the same sheaf meet, but each meets each cubic of the other sheaves once. In six of the nine sheaves, each cubic meets one triple line twice, each of the other two triple lines once and each double line once; in the remaining three sheaves, each cubic meets two triple lines twice, the third triple line once, one double line once and the remaining two double lines not at all. If we count the triple lines as isolated cubics, each cubic of the nine sheaves meets 70 isolated cubics once and the remainder twice.

Each isolated cubic meets five lines once while it fails to meet the other four. Including the double lines, it meets six conics twice, 20 conics once and ten not at all. It meets each cubic of four sheaves twice and each cubic of the remaining five sheaves once. It meets five of its kind thrice, 40 twice and 60 once, while it fails to meet the remaining 20. As will be noted below, in eleven cases the residue quartic breaks up into a triple line and a simple line. The relations of these cubics to the multiple lines of the surface will be more easily pointed out when we come to consider the complete sections.

There are no plane cubics other than those thus enumerated.

Plane quartics can only exist in three ways: (a) such as complete the sections of the planes of the above-noted cubics; (b) those that are the residues of planes having the triple lines as axes, and (c) such as are ingredients of degenerate cases of plane quintics and sextics. Of (a) there are 123 isolated cases imaged as conics through four fundamental points, their fifth points being determinate points, and nine sheaves imaged as non-singular cubics through eight fundamental points; of (b) there are three sheaves imaged the same as the 123 isolated cases of (a) except that their fifth points are undetermined in the projection; while of (c) there are, as we shall see below, six cases that are the part ingredients of degenerate quintics of the sheaves through the double lines, but which, however, are identical with six of the quartics of the nine sheaves of (a). So that, altogether, we have three sheaves and 123 isolated examples of unicursal plane quartics and nine sheaves of plane quartics, $d = 1$.

Each quartic of the nine sheaves meets eight simple lines once, while it fails to meet the ninth. This is as it should be, since the general plane section meets

each line once, and all degenerate examples of such sections must preserve this property; and as we saw above that each cubic of the nine sheaves met one line and failed to meet the other eight, it becomes evident that the quartic associated to that cubic must meet those eight once and fail to meet that one. However, as we have already noted that in eight cases in each sheaf the cubic breaks up into a line and a conic, it is plain that in each sheaf there are eight quartics, each of which meets one line four times and one conic eight times.

In each sheaf there is one quartic that breaks up into a cubic and a line; we have already referred to this case when considering the associated cubics. It is that quartic of the sheaf whose projected cubic, passing through the eight fixed points of the sheaf, still passes through the ninth point, which point, as we know, is the center of the sheaf of lines that represent the associated cubics. This quartic, therefore, must break up into the cubic that is imaged by the cubic through the nine fundamental points and into the line imaged by the ninth point.

Except in the 72 cases noted above, however, each quartic of these nine sheaves, harmonious to its associated cubic, meets eight conics twice and the remainder once; while, counting the triple lines as isolated cubics, it meets 70 such cubics twice and the remainder once. Each such quartic, moreover, meets each of its companions of the same sheaf once, and of the other sheaves twice.

Moreover, each such quartic meets its associated cubic three times off of the multiple lines and, consequently, meets it nine times upon these lines. The cubic is unicursal, the quartic of deficiency one, and the section must possess three double and three triple points. Therefore, the intersection of these two curves must be such that the cubic has its node upon one triple line and the quartic its two nodes upon the other two triple lines; each curve passes through the node or nodes of the other and both still intersect each other upon the three double lines; the three remaining points of intersection are, consequently, points of contact of the plane of the section, and we thus obtain, at once, nine sheaves of tritangent planes. Moreover, there are 12 of such planes in each sheaf that are quartuply tangent; for Cayley, in the fourth volume of Crelle, p. 167, shows that the number of curves of a sheaf of n^{thic} curves possessing α common double points, which still possess another double point, is

$$3(n-1)^2-7\alpha;$$

this, for $n=3$, $\alpha=0$, gives 12. We thus obtain (since this additional double

point can be nothing else than the image of a point of contact of the plane containing the imaged quartic) 72 quarti-tangent planes.

Each of the 123 isolated quartics meets four lines once and fails to meet the remainder. Including the double lines, it meets 10 conics twice, 20 once and fails to meet the other six. It meets the cubics of five sheaves twice and those of the remaining four sheaves once. Of the associated quartics of these sheaves, it meets those of four sheaves thrice and those of five sheaves twice. It meets five of its own kind four times, 40 three times, 60 twice and 20 once, counting the special quartics in the planes through the triple lines. As was noted above and as will be shown below, in eleven cases these quartics break up into a triple line and a simple line.

Each of these quartics meets its associated cubic four times off of the multiple lines and, therefore, meets it eight times upon them. Since both curves are unicursal and since the section must have three double and three triple points, there are two classes of intersections, and the planes are so classified. (a). The quartic has its nodes upon the three triple lines; the cubic passes through these nodes, intersects the quartic still upon two double lines and has its node upon the third double line; there are, therefore, three varieties of this case. (b). The quartic has two of its nodes upon two of the triple lines, upon the third of which the node of the cubic lies; the two curves each pass through each other's node or nodes that lie upon the triple lines and intersect still upon two of the double lines, upon the third of which the third node of the quartic lies; there are nine varieties of this case. The four residue points of intersection of the two curves are, consequently, points of contact, and we thus obtain 123 new quarti-tangent planes.

Finally, we have the quartics of the three sheaves of planes through the triple lines, imaged as conics through four fundamental points. We have already seen that each triple line is met by five lines in such a manner that four of these fifteen lines are common to A_1 and B_1, one common to A_1 and C_1 and one common to B_1 and C_1, leaving the necessary remainder of nine. In such planes, therefore, the quartic breaks up into a cubic and a line, the cubic being imaged as a conic through five points; this fifth point, being a point of its intersection with the image of the axial triple line, is the image of the residue line. In four of these 15 planes, however (the four though, in fact, being but two counted twice), the plane becomes still further specialized, in that the cubic itself degene-

rates into a triple line; that is, the conic that images this cubic becomes identical with the conic that images the triple line in question. This is as it should be, since we know that A_1 and B_1 each meets C_1 and determine planes whose residues must be simple lines of the surface; these two planes, therefore, count once in each of the sheaves through A_1 and B_1 and twice in the sheaf through C_1. The remaining 11 planes are identical with the 11 cubo-quartic planes already referred to, in which the quartic breaks up into a triple line and a simple line.

Further, we have already seen that, including the double lines, each triple line is met twice by six conics; such planes, therefore, are determinate planes, in which the residue quartic breaks up into two conics. These cases, however, have already been treated in the discussion of the group of conics and notice taken that, since A_1 and B_1 each meet each double line once, while C_1 meets none, three conics group themselves with the double lines to form conic-pairs in the sheaf through A_1, three more similarly in the sheaf through B_1, while the remaining six group themselves in pairs in the sheaf through C_1, nine planes (three in each sheaf) being thus determined; in six of these planes, then (three through A_1 and three through B_1), the quartic breaks up into a double line and a conic and in the remaining three (those through C_1) it breaks up into two undegenerate conics.

The relations that these quartics bear to the simple lines, conics, plane cubics and other plane quartics of the surface, are the same as those already noted in the case of the 123 isolated quartics of the surface.

The quartics of the sheaves through A_1 and B_1 have a triple point that lies, in the one case, upon B_1 and, in the other case, upon A_1, while the axis meets C_1 and the three double lines. Since the section is thus complete and the quartic meets its axis four times, all such planes have triple contact along each of four sheets of the surface.

The quartics of the sheaf through C_1 are trinodal, having a node upon each of the double lines, while the axis meets A_1 and B_1; the section being thus complete and the quartic meeting its axis four times, these planes also have triple contact along each of four sheets of the surface. So that, altogether, the surface possesses three sheaves of planes of this kind.

We have still to examine the special planes of these sheaves.

The six planes, each of whose sections consists of a triple line, a double line and a conic, have a sextuple contact along one sheet of the surface, triple contact

along two sheets and double contact along a fourth. This can best be seen by an example. Let the section be imaged by A_1', D_1' and 67. This last meets A_1' twice, but only meets D_1' once ulterior to the fundamental points; but since the conic imaged by 67 lies in the same plane with D_1, it must meet D_1 twice; one intersection must have been, therefore, absorbed in the projection of the multiple lines; but the only multiple line, besides A_1, that is met by D_1 is B_1; consequently, one point of intersection of the conic with D_1 must be at the point where D_1 meets B_1; in fact, when we consider the complete section, we see that this must be so. The axis A_1 meets C_1, E_1 and F_1, while D_1 and the conic meet B_1; this completes the section and leaves the intersection of A_1 and D_1, the two intersections of A_1 and the conic and the remaining intersection of D_1 and the conic to be reckoned as points of contact. We thus obtain the species of contact noted in the first sentence of this paragraph.

The three planes, whose sections each consists of a triple line and two undegenerate conics, can be, in the same manner, shown to have triple contact along four sheets of the surface and ordinary contact along a fifth sheet. C_1, the axis, meets both A_1 and B_1, while the two conics intersect upon the double lines; this leaves, for points of contact, the fourth point of intersection of the two conics and the four points in which the conic meets C_1. We see that three points of intersection of the two conics are absorbed in the projection of the double lines, as is also evident from the plane representation.

The two planes, whose sections each consists of two triple lines and a simple line, have contact still more highly singular. In both cases, C_1 is an ingredient and, therefore, meets the third triple line, while the second triple ingredient meets the three double lines and thus completes the section. The intersections of the three ingredients thus remain as points of planar contact. Each of these planes, therefore, has a 9-tuple contact along one sheet of the surface and triple contact along two other sheets.

There remains, to be examined, those planes, whose sections of the surface each consists of a triple line, a simple line and a nodal cubic, four of which occur in each of the sheaves through A_1 and B_1 and three in the sheaf through C_1.

The first eight planes have triple contact along four sheets of the surface and ordinary contact along a fifth sheet. An example will plainly reveal this. Let the section be imaged by 1 2 3 4 5, 4 6 7 8 9, 4. The axis A_1 meets

C_1, D_1, E_1, F_1, accounting for all the multiple points of the section with the exception of the triple point upon B_1; as A_1 does not meet B_1, the node of the cubic must lie upon B_1 at the point where the line of the section meets B_1 (this is also evident from the plane representation); there remains, consequently, for points of planar contact, the intersections of A_1 with the cubic and the simple line and the remaining intersection of these last two.

The three remaining planes can be, in the same manner, shown to have a triple contact along four sheets and ordinary contact along a fifth sheet. Of such singular planes we have, therefore, altogether fourteen examples.

We have now to consider the plane quintics, of which 21 isolated examples complete the sections through the 21 unabsorbed conics, while the remainder are the residues of the three sheaves of planes through the double lines.

The planes of the 21 quintics are tritangent planes and are of two kinds. The quintics, being of deficiency one, have five double points or one triple point and two double points. In the one species of plane, the quintic has three nodes upon the triple lines and the two nodes upon the two double lines; the conic passes through the nodes upon the triple lines and intersects the quintic still again upon the third double line. There are, consequently, three varieties of this species of plane. In the second species of plane, the quintic has a triple point upon one triple line and double points upon the other two triple lines; the conic passes through the two double points and still intersects the quintic upon each of the three double lines. There are, then, also three varieties of this species of plane.

Each one of these isolated quintics meets seven lines and fails to meet the other two; including the triple lines, it meets 21 of the isolated cubics once, 70 twice, and 35 three times, while it meets the associated quartics analogously; it meets each cubic of seven sheaves twice and of two sheaves thrice, while it meets the associated quartics analogously; it meets the quartics of the three sheaves in exactly the same manner as it meets the other unicursal quartics.

The quintics of the three sheaves have precisely the same relations to the simple lines and plane curves of the surface as the quintics just discussed. There are, however, to each sheaf, two planes in which the quintic breaks up into a line and a quartic. These six planes have already been remarked, in the discussion of the plane quartics, as planes in which the cubic broke up into a

conic and a line. Such quartics are, consequently, common to the sheaves in question.

These special planes are planes having double contact along three sheets of the surface and ordinary contact along a fourth sheet, while the general planes of the sheaves each has double contact alone along three sheets of the surface. The contact takes place as follows: In the general plane, the axis meets both A_1 and B_1; through these points the quintic passes, has a triple point upon C_1 and two double points upon the two double lines; the remaining three points of intersection of the axis and the quintic are points of double contact. In the six special planes, the axis meets A_1 and B_1; the quartic passes through these points and has one double point upon C_1 and another double point upon one of the two remaining double lines; the residual line passes through the node upon C_1 and meets the quartic again upon the second remaining double line; the two residue intersections of the axis and the quartic, the intersection of the axis and the line and the remaining intersection of the quartic and the line are thus left as points of double and simple contact.

Of plane sextics, there are nine sheaves of deficiency, $d = 2$, being the residues of the sheaves of planes through the simple lines of the surface; such sextics are imaged as nodal quartics through the nine fundamental points with the node at the image of the axial line. These planes are of two kinds. In five of the nine sheaves, the section is of such sort that the sextic has a triple point upon one triple line and double points upon the other multiple lines; the line of the section passes through the two double points of the sextic that lie upon the triple lines. In the remaining four sheaves, the section is of such sort that the sextic has a triple point upon A_1 and B_1 and a double point upon C_1 and one double line, while the line of the section passes through the node at C_1 and still intersects the sextic upon the two remaining double lines. In both of these kind of planes, therefore, the line still meets the sextic twice, the points of intersection being the two points of contact of the plane (imaged by the node of the imaging quartic). This quartic may, however, have another double point, in which case the sextic of the surface has another node and the plane becomes tritangent; in each sheaf, therefore, there are 20 tritangent planes, giving, altogether, 180 new planes of this sort.

Each sextic of these sheaves meets the eight remaining lines once; counting the double lines, it meets eight conics once and the remainder twice, while it

meets the associated quintics analogously; it meets the cubics of eight sheaves thrice and those of the other sheaf twice and the associated quartics analogously; it meets 70 of the isolated cubics twice and the remainder three times, and it meets the associated quartics analogously. Each sextic of the sheaf meets each of its kind of its own sheaf six times and, of the other sheaves, five times.

Finally, we have a quintuply-infinite sheaf of plane septics, of which all the above-enumerated curves form degenerate examples. These curves are the curves of the general plane section, and their properties have already been remarked.

Of twisted curves we have, upon the surface, of order not greater than six, the following: Nine sheaves of twisted cubics, imaged as lines through one fundamental point, and 126 isolated twisted cubics imaged as conics through five fundamental points; one net, 126 sheaves and 162 isolated examples of twisted quartics of the second kind, imaged, respectively, by the arbitrary lines of the plane, the conics through four fundamental points and the nodal cubics through seven; nine sheaves of twisted quartics of the first kind, imaged by non-singular cubics through eight fundamental points; 84 nets, 504 sheaves and 513 isolated examples of twisted quintics, $d = 0$, imaged, respectively, by conics through three points, nodal cubics through six points, and by 504 trinodal quartics through eight points and nine triple-pointic quartics through nine points; 36 nets and 36 sheaves of twisted quintics, $d = 1$, imaged, respectively, by non-singular quartics through seven and by binodal quartics through nine fundamental points; 36 webs, 630 nets, 1332 sheaves and 756 isolated examples of twisted sextics, $d = 0$, imaged, respectively, by conics through two points, nodal cubics through five points, 1260 trinodal quartics through seven and 72 triple-pointic quartics through eight points and 504 triple-pointic and trinodal quintics through nine and 252 6-nodal quintics through eight points; 84 webs, 288 nets and 126 sheaves of twisted sextics, $d = 1$, imaged, respectively, by non-singular cubics through six, binodal quartics through eight and 5-nodal quintics through eight points; and nine webs of twisted sextics, $d = 2$, imaged by nodal quartics through nine fundamental points. There are no twisted quintics of deficiency greater than one and no twisted sextics of deficiency greater than two upon the surface.

The twisted cubics meet the simple lines, the multiple lines and the various plane curves of the surface in the same manner as the plane unicursal cubics do; e. g. four sheaves of the twisted cubics meet A_1 and B_1 once, C_1 twice and the

double lines each once; those of one sheaf meet A_1 twice and the other multiple lines each once; those of the sixth sheaf meet B_1 twice and each of the other multiple lines once, while those of the remaining three sheaves meet A_1 and B_1 twice, C_1 once and either D_1, E_1 or F_1 once, respectively.

Each quartic of the net fails to meet any line of the surface and, therefore, meets each associated sextic four times; it meets each conic once and the associated quintic and each quintic of the three sheaves, analogously; it meets each cubic of the nine sheaves once and the associated quartic analogously; it meets each isolated cubic twice and the associated quartic and each quartic of the three sheaves analogously; it meets each triple line twice and each double line once; and it meets the twisted cubics in the same way as it does their plane correspondents. Any two of these quartics meet once.

The other twisted quartics of the surface meet all other curves of the surface, just as the plane quartics, of the same species, do.

Each quintic of the 84 nets meets three lines once and fails to meet the remainder and, therefore, meets the plane sextics of three sheaves four times and those of six sheaves, five times; counting the double lines, it fails to meet three conics, meets 18 once and the remainder twice, while it meets the associated quintics and the quintics of the three sheaves analogously; it meets each of the cubics of three sheaves once and those of the remaining six sheaves twice and, consequently, meets the associated quartics analogously; counting the triple lines, it meets 15 of the isolated cubics once, 60 twice, 45 thrice and the remainder four times, while it meets the associated quartics and the quartics of the three sheaves analogously. Each triple line is met once by each of the quintics of 10 nets, twice by each of those of 40 nets, thrice by each of those of 30 nets and four times by each of those of the remaining four nets. It meets each twisted quartic of the net twice, and the other twisted curves, previously mentioned, are met by it in the same manner as their correspondents in the plane sections are. Each quintic of these 84 nets meets each of its own kind as follows: it meets each of those of the same net once, each of those of 18 nets twice, each of those of 45 nets thrice and each of those of the remaining 20 nets, four times.

Each quintic of the 504 sheaves meets one line twice, five lines once and fails to meet the remaining three; it meets the associated sextics analogously; counting the double lines, it meets three conics thrice, 15 twice, 13 once and fails to meet the remaining five; it meets the associated quintics and the quin-

tics of the three sheaves analogously; it meets each of the cubics of one sheaf once, each of those of five sheaves twice, and each of those of the remaining three sheaves thrice; it meets the associated quartics analogously; counting the triple lines, it meets ten of the isolated cubics four times, 35 thrice, 45 twice, 31 once and fails to meet the remaining five; it meets the associated quartics and the quartics of the three sheaves analogously; it meets each twisted quartic of the net three times; it meets each twisted quintic of ten nets twice, each of those of 25 nets thrice, each of those of 33 nets four times, each of those of 15 nets five times, and each of those of the remaining net six times; it meets all other previously mentioned twisted curves in the same manner in which it meets their correspondents in the plane sections. Each triple line is met not at all by each of the quintics of 20 sheaves, is met once by each of those of 124 sheaves, is met twice by each of those of 180 sheaves, is met thrice by each of those of 140 sheaves, and is met four times by each of those of 40 sheaves. Each double line is met not at all by each of the quintics of 70 sheaves, is met once by each of those of 182 sheaves, is met twice by each of those of 210 sheaves, and is met thrice by each of those of 42 sheaves. Each quintic of these 504 sheaves meets each of its own kind, of its own sheaf, twice and each of those of the other sheaves from three to seven times, according to the number of coordinated lines they have in common.

The remaining loci of the surface and their configuration with the other curves of the surface can be easily calculated along the lines already followed.

The general plane section of *the surface*, $S^{(7)}_{1^3,\,5^2}$, consists of a septic curve possessing one triple and five double points and is imaged, in the simplest plane representation, by a septic curve which passes through 19 fixed points of which one is triple and five are double for the curve; these are the fundamental points of the projection. This surface possesses, then, only the 13 lines imaged by the 13 simple fundamental points and which, therefore, must be double chords of the double quintic that still meet the triple line; this is also apparent from the plane representation. These lines do not meet each other.

Since every plane through the triple line cuts out a quartic curve and since there is only one sheaf of plane quartics, imaged by the lines of the plane that center at the triple fundamental point, the image of the triple line must be the sextic curve, $1^2 . 2^2 . \dots . 6^2 . 7 . 8 \dots . 19$, and, consequently, the image of the double quintic must be the 12^{thic}, $1^6 . 2^3 . 3^3 \dots . 6^3 . 7^2 . 8^2 \dots . 19^2$, where 1 repre-

sents the triple fundamental point, 2 .... 6 represent the double fundamental points and 7 .... 19 represent the simple fundamental points.

The five double fundamental points and the five lines that join the triple to the double fundamental points image the ten conics possessed by the surface; these conics, consequently, occur in pairs, being, in fact, degenerate cases of the plane quartics of the sheaf. These conics do not meet the lines of the surface, but are each met by the double quintic thrice and each, of course, meets the triple line twice.

Since the conics occur in pairs as degenerate cases of the quartics of the sheaf, and since the surface possesses no other plane quartics than these, the plane cubics possessed by the surface can only occur in the 13 planes determined by the triple line and a simple line, being the residual sections of the surface by such planes; such planes, therefore, are degenerate planes of the sheaf of planes through the triple line, in which the residual quartics break up into lines and cubics. These 13 plane cubics are imaged, then, by the lines of the plane that join the triple fundamental point to the simple fundamental points.

Since no two lines of the surface meet, and since all conics occur in pairs, the surface can possess no plane quintics.

Of plane sextics, however, there are 13 sheaves, being the residual sections of the surface by the planes having for axes the simple lines of the surface; such are, consequently, imaged by septic curves through all the fundamental points, having triple points at the triple point, double points at the double points, and a sixth double point at the image of the axial line. In each sheaf, one plane breaks up into the triple line and a cubic, being the planes already mentioned.

These curves, with the septics of the general plane section, complete the list of plane curves possessed by the surface. Of such septics, there is a single sheaf.

Each cubic of the 13 planes fails to meet any line except the one coordinated to itself, and fails to meet any other plane curve except the sextics, each of which it meets three times. It meets the double quintic four times.

No plain quartic meets a line nor a plane curve except the sextics, each of which it meets four times. It meets the double quintic six times.

Each plane sextic meets each of the residue 12 lines once, each conic twice, each cubic thrice and each quartic four times. Two sextics of the same sheaf

meet four times, of different sheaves five times. Each sextic meets the double quintic eight times and the triple line four times.

Each sextic and its associated line meet twice off of the multiple curves and four times upon them; the sextic has a double point upon the triple line and three more upon the double quintic; the line passes through the node upon the triple line and meets the sextic still twice upon the double quintic. Of these 13 sheaves of bitangent planes, there will be, after the formula of Cayley, 312 that are tritangent.

The general planes of the sheaf of planes through the triple line have triple contact along four sheets of the surface since the triple line meets the double quintic twice and the quartic residue has its three nodes upon the double quintic; the four points of intersection of the triple line and the quartic are points of planar contact.

In the 13 special planes of this sheaf in which the quartic degenerates into a line and a cubic, the triple line meets the double quintic twice, as before, while the nodal cubic has its node upon the double quintic and still meets the associated line twice upon that curve; the intersections of the triple line with the simple line and with the cubic and the third intersection of these last two with each other are all points of planar contact; these 13 planes, thus, have triple contact along four sheets of the surface and ordinary contact along a fifth.

The five planes in which the quartic degenerates into a pair of conics have, similarly, triple contact along four sheets and ordinary contact along a fifth. There are, therefore, altogether 18 planes which possess this order of singular tangency.

Of the twisted curves upon the surface, the rational twisted cubic which is imaged by the triple fundamental point is a distinguished one. This cubic fails to meet any line of the surface and only meets one conic of each pair. It meets each plane cubic and each plane quartic each once and meets each plane sextic twice. It meets the triple line twice and the double quintic six times.

Besides this distinguished cubic, the surface still possesses 15 other twisted cubics, of which 10 are imaged by the lines that join any two of the double fundamental points and the remaining five by conics passing through the triple and any four of the five double fundamental points. These 15 cubics behave in the same manner as the one already discussed.

The surface possesses 65 twisted quartics of the second kind, imaged by lines

joining the double to the simple fundamental points, and one sheaf of such quartics imaged by the lines of the plane that center at the triple fundamental point. It still possesses 144 isolated examples of such quartics, of which one is imaged by the conic determined by the five double fundamental points, 130 imaged by conics determined by the triple, three double and one simple fundamental point and 13 imaged by cubics through all the multiple and one simple fundamental point having their nodes at the triple point.

The surface does not possess any twisted quartic of the first kind.

Of twisted quintics, $d=0$, the surface possesses 16 sheaves and 1378 isolated examples. Five sheaves are imaged by the sheaves of lines of the plane that have, for centers, the five double fundamental points; 10 sheaves are imaged by the conics of the plane that pass through the triple and any three double fundamental points; and the remaining sheaf is imaged by the nodal cubics that, having a node at the triple point, still pass through the five double points. Of the 1378 isolated examples, 78 are imaged by the lines that join the simple fundamental points, two by two; 65 are imaged by the conics that pass through four double and one simple fundamental point; 780 are imaged by the conics that pass through the triple, two double and two simple fundamental points; 65 are imaged by the cubics that, having a node at one double point, still pass through the remaining multiple points and one simple point; and 390 are imaged by the cubics that, having a node at the triple point, still pass through four double and two simple fundamental points.

Of twisted quintics, $d=1$, the surface possesses 286 isolated examples which are imaged by non-singular cubics determined by all the multiple and three simple fundamental points.

The surface possesses no twisted quintic of deficiency higher than one.

Of the 209 twisted quartics possessed by the surface, all but one each meet one line of the surface. This one, being the distinguished quartic imaged by the conic determined by the five double fundamental points, fails to meet any line of the surface, but meets each conic once, each plane cubic and each plane quartic twice, each plane sextic four times, fails to meet any twisted cubic and meets each other twisted quartic once. Of the remaining 208 twisted quartics, each of 65 meets four conics twice, two conics once and fails to meet the remainder and fails to meet any plane cubic; each of the residue 143 meets five conics once, fails to meet the other five, fails to meet one plane cubic and meets

the other 12 each once. Each of these 208 quartics meets each quartic of the sheaf once, each sextic of 12 sheaves four times and each sextic of one sheaf three times; it meets 60 of its kind twice, 130 once and 18 twice; while each of 65 meets one twisted cubic twice, 10 once and fails to meet five, at the same time that each of 143 meets two twice, eight once and fails to meet six. Each of these quartics meets the triple line twice and the double quintic seven times, except that the distinguished quartic meets this last curve nine times.

No quintic of the sheaves meets a line of the surface. Each one meets one conic of each pair once but fails to meet its companion. Each meets each plane cubic once, each plane quartic once and each plane sextic five times. Each quintic of five sheaves meets one twisted cubic twice, 10 once and fails to meet five; each quintic of 10 sheaves meets one twisted cubic twice, eight once and seven not at all; while each quintic of the sixteenth sheaf meets every twisted cubic once. Each quintic of the five sheaves fails to meet 13 of the 65 twisted quartics but meets the remaining 52 once; it meets the distinguished quartic once, 79 of the 130 once and 52 twice, and it meets the 13 twice; each quintic of the 10 sheaves meets 39 of the 65 once and 26 twice; it meets the one once, it meets 39 of the 130 twice, 78 once and fails to meet 13, and it meets the 13 twice; each quintic of the sixteenth sheaf meets each of the 65 twisted quartics twice, meets the one once, each of the 130 once and fails to meet the 13.

Of the 1378 isolated twisted quintics, $d=0$, 1248 each meet two lines, each once, and 130 each meet one line once. Each of these same 1248 meets one conic from each pair, meets 11 cubics once, failing to meet the others, meets each quartic of the sheaf once, meets each sextic of 11 sheaves four times and each sextic of two sheaves three times; on the other hand, each of the above-mentioned 130 meets each conic of four pairs but fails to meet one conic of the fifth pair, meeting its companion twice; it meets 12 plane cubics twice and the thirteenth once; it meets each quartic of the sheaf twice; it meets each sextic of 12 sheaves five times and each sextic of the thirteenth sheaf four times. Each of the 1248 meets five twisted cubics twice, 10 once and fails to meet the sixteenth; it meets 11 twisted quartics thrice, 113 twice, 75 once and fails to meet 10; it meets each twisted quintic of one sheaf thrice, of 10 sheaves twice and of five sheaves once; on the other hand, each of the 130 meets eight twisted cubics once and fails to meet the other eight; it meets 96 twisted quartics twice, 104 once and fails to meet nine; it meets each quintic of eight sheaves twice and meets

each quintic of the other eight sheaves once. Each quintic of these 1378 twisted quintics meets each other one once, twice or thrice according to their mutual relations to the lines and conics of the surface.

Each twisted quintic, $d = 1$, meets three lines of the surface, meets every conic, meets 10 plane cubics twice and three once, meets each quartic of the sheaf twice, meets the sextics of three sheaves four times and of 10 sheaves five times; it meets each of the twisted cubics once, 112 twisted quartics twice and 97 once, each quintic of the 16 sheaves twice, 720 of the 1378 isolated twisted quintics thrice, 580 twice and 78 once. Each of these 286 twisted quintics meets 30 of its kind once, 135 twice and 120 thrice.

The remaining loci of the surface and the relations that exist between themselves and between them and the other surface curves can be easily discussed along the lines already followed.

In *the surface*, $S^{(7)}_{2^3,\,4^2}$, the general sections consist of septic curves possessing two triple and four double points, imaged, in the simplest plane representation, by sextic curves, having in common five double and nine simple fundamental points.

These nine simple points image the only lines possessed by the surface; these lines are all non-planar, one being a chord of the triple conic and the other eight being chords of the double quartic that still meet the triple conic.

There are 16 simple conics upon the surface; five of these are imaged by the double fundamental points, 10 by the lines that join these points, two by two, and the sixteenth by the conic determined by these points. The first 15, consequently, form triplets or quadruplets in such a manner that while neither the first five meet each other nor the last 10 meet each other, these last are so associated to the first five that each of these five meets four of the 10 while each of the 10 meets two of the five; these 15 conics can be therefore grouped into 10 triplets or five quadruplets. The sixteenth conic meets each of the first five but fails to meet any of the last 10. Moreover each one of the 10-let joins itself to each one of two out of the 5-let to form a pair that is distinguished as being a degenerate example from the sheaves of trinodal twisted quartics that are imaged by the lines of the representative plane that center at the double fundamental points, four of such examples occurring in each sheaf. None of these simple conics meets a line of the surface.

The triple conic is imaged by the sextic $1^2 \ldots\ldots 5^2 . 6^2 . 7 \ldots\ldots 14$, where 6

is the image of the chord of the triple conic and the double quartic is imaged by the hyperelliptic 9^{thic}, $1^3 \ldots . 5^3 . 7^2 \ldots . 14^2$. So that, as stated above, the remaining eight lines are the double chords of C_4 that still meet C_2.

Each of the simple conics meets the triple conic twice and the double quartic thrice.

There are no plane cubics or quartics upon the surface.

Since there are no line pairs upon the surface, the only plane quintics that the surface can possess are the 16 that are the residue sections of the planes through the 16 conics. Five of these are imaged by sextic curves through the 14 fundamental points having a triple point at the image of its associated conic and double points at the other four double fundamental points; 10 are imaged by quintic curves through all the fundamental points with double points at three of the double points; and the sixteenth quintic is imaged by the non-singular quartic that is determined by the 14 fundamental points. Each of these quintics is met by each line once.

The nine sheaves of plane sextics, whose planes have for axes the lines of the surface, are imaged by 6-nodal sextics through all the fundamental points with nodes at the five double and one simple fundamental point; in the sheaf through the line whose image is 6, one sextic degenerates into the triple conic of the surface. Each sextic meets each of eight lines once and each conic twice.

The configuration of the plane quintics among themselves is analogous to that of the conics among themselves. Two sextics of the same sheaf meet four times; of different sheaves, five times.

Of the plane septics of the general plane section, there is one web.

The nine sheaves of planes through the lines of the surface are, ordinarily, bitangent. The line meets the curve twice off of the multiple curves and four times upon them; the sextic is of deficiency four, while, in eight sheaves, the line is a double chord of the double quartic still meeting the triple conic. In such sheaves, therefore, the sextic has a triple and a double point upon the triple conic and two double points upon the double quartic, while it meets the line upon the double conic at the double point and twice again upon the double quartic; the residue intersections of the line and the sextic are points of contact; in each sheaf there are 33 planes that are tritangent. In the ninth sheaf the sextic is 6-nodal; two nodes, through which the line passes, are upon the

triple conic and the other four upon the double quartic; in this sheaf there are also 33 tritangent planes, making altogether, so far, 297 tritangent planes.

The planes of the conics and the quintics, which curves meet seven times upon the multiple curves and three times off of them, are tritangent; the quintics, being of deficiency three, have two double points upon the triple conic and one upon the double quartic; the associated conic passes through the nodes upon the triple conic and still meets the quintic thrice upon the quartic; the three remaining points of intersection of the conic and the quintic are points of planar contact. This makes, altogether, 313 tritangent planes.

The plane of the triple conic has triple contact along two sheets of the surface.

There are 90 isolated twisted cubics upon the surface, of which 45 are imaged by the lines joining the double to the simple fundamental points and 45 by the conics determined by four double and one simple fundamental point. This completes the list of cubics possessed by the surface.

There are 10 sheaves and 576 isolated examples of twisted quartics of the second kind. Five sheaves are imaged by the lines that center at the double fundamental points and five by the conics that pass through four double fundamental points; four times in each sheaf, a quartic breaks up into a pair of non-planar conics and nine times into a non-planar cubic and a line. Thirty-six of the isolated quartics are imaged by the lines that join the simple fundamental points two by two; they are special examples out of sheaves of quintics; 360 are imaged by conics passing through three double and two simple fundamental points, and are also special examples out of sheaves of quintics; and 180 are imaged by nodal cubics through the five double and two simple fundamental points, node at a double point.

There are 126 isolated twisted quartics of the first kind, imaged by non-singular cubics through all the double and four simple fundamental points.

Of twisted quintics, $d = 0$, there are 144 sheaves and 3452 isolated examples. Nine sheaves are imaged by the lines that center at the simple fundamental points, 90 sheaves by the conics through three double and one simple fundamental point and 45 sheaves by nodal cubics passing through all the double and one simple fundamental point. Of the isolated quintics, 840 are imaged by conics through two double and three simple fundamental points; 1680 by nodal cubics through four double and three simple fundamental points, with node at a

double point; 72 by nodal cubics through all the double and two simple fundamental points, having the node at the simple point; and 840 by trinodal quartics through all the double and three simple fundamental points, with the three nodes at three double points.

Of twisted quintics, $d = 1$, there are 84 sheaves, imaged by non-singular cubics through all the double and three simple fundamental points and 1890 isolated examples, of which 630 are imaged by non-singular cubics through four double and five simple fundamental points and 1260 are imaged by binodal quartics through the five double and five simple fundamental points.

Of twisted quintics, $d = 2$, there are 180 isolated examples, imaged by uninodal quartics through all the double and seven simple fundamental points, node at a double point.

A distinguished twisted curve of the surface is the 9^{thic} imaged by the cubic determined by the nine simple fundamental points.

Each twisted cubic meets some one line of the surface once. Each of the first 45 meets 12 conics once and fails to meet the other four, while each of the second 45 meets each of eight conics once and fails to meet the other eight. Each of the first 45 meets each of 12 plane quintics twice and each of four thrice, while each of the second 45 meets each of eight plane quintics twice and each of the other eight thrice. Each of these 90 twisted cubics meets the sextics of one sheaf twice and those of the remaining eight sheaves thrice; the sextics that are met but twice have for their axis the line that is met by the cubic considered. Each twisted cubic of each set of 45 meets 32 of its own set once and fails to meet the other 12, while it meets, of the other set, eight twice, 33 once, and four not at all. Each twisted cubic meets the double quartic four times, while 80 meet the triple conic thrice, the other 10 meeting it twice.

Each quartic of the 10 sheaves of twisted quartics fails to meet any line of the surface; it meets eight conics once and fails to meet the other eight; it meets the plane quintics and the plane sextics in a manner analogous to the way in which it meets, respectively, the conics and the lines; it meets the triple conic four times and the double quartic six times; it fails to meet any of its own sheaf but meets each quartic of eight sheaves once and each quartic of the remaining sheaf twice.

Each quartic of the 576 isolated quartics, $d = 0$, meets some two lines of the surface; it meets one conic twice, 10 once and fails to meet five; it meets

the plane quintics and the plane sextics of the surface in a manner analogous to the way in which it meets, respectively, the conics and the lines; the triple conic is met by 128 of these quartics three times, and is met four times by the remainder, while each of the 576 quartics meets the double quartic five times; each quartic of these isolated quartics meets 10 twisted cubics twice, 70 once and fails to meet the remaining 10; it meets each twisted quartic of five sheaves twice and each one of the other five sheaves once; and, finally, it meets, of its own kind, 105 thrice, 280 twice, 166 once and fails to meet 24.

Each of the 126 twisted quartics, $d = 1$, meets four lines of the surface; it meets each conic once; it meets the plane quintics and the plane sextics in a manner analogous to the way in which it meets, respectively, the conics and the lines; it meets 50 of the 90 twisted cubics twice and 40 once; it meets each quartic of the 10 sheaves twice; it meets, of the isolated twisted quartics, $d = 0$, 160 thrice, 320 twice and 96 once. Each of these 126 twisted quartics, $d = 1$, meets, of its own kind, 5 four times, 20 thrice, 60 twice and 40 once. It meets the double quartic four times, while the triple conic is met four times by 70 and three times by the remaining 56.

Each quintic of the 144 sheaves of twisted quintics, $d = 0$, meets a single line of the surface while, of the 16 conics, it meets one twice, 10 once and fails to meet five; it meets the plane quintics and the plane sextics of the surface in a manner analogous to the way in which it meets, respectively, the conics and the lines of the surface; each quintic of the nine sheaves meets 80 twisted cubics once and fails to meet 10; each quintic of the 90 sheaves meets 32 twisted cubics twice, 52 once and fails to meet six, and each quintic of the 45 sheaves meets 64 twisted cubics twice, 24 once and fails to meet two; each quintic of these 144 sheaves meets each twisted quartic of five of the 10 sheaves twice and each one of the other five sheaves once; each of these quintics meets 140 of the isolated twisted quartics, $d = 0$, thrice, 320 twice, 108 once, while it fails to meet eight; it meets, of the 126 isolated twisted quartics, $d = 1$, 70 thrice and 56 twice; it meets the double quartic seven times, while the triple conic is met by the quintics of three sheaves four times and by each of those of the remaining sheaves five times; and, finally, each quintic of each sheaf fails to meet any quintic of its own sheaf, but of the quintics of the remaining 143 sheaves, it meets each one of 40 sheaves thrice, each one of 85 sheaves twice and each one of 18 sheaves once.

The relations borne by the remaining enumerated twisted curves, as well as the number, configurations and relations of those not enumerated, can be easily calculated and expressed by a continuation of the processes already employed.

In conclusion, it is only necessary to state that the ordinary singularities of these surfaces, such as the class and rank, the number of inflexional tangents, the class of the developable, the number of double tangents, the class of the spinode-torse and of the node-couple-torse, etc., are readily calculated from the general formulæ.

COLUMBIA UNIVERSITY.